Abstract – Return series to broad asset classes often possess histories of unequal length as well as the presence of smoothing. Covariances typically revert to using the common, though shorter, series length while covariances for smoothed series are necessarily biased downward. These pose serious problems which, if left unresolved, will generate suboptimal and misleading allocations across asset classes. This paper discusses and draws together elements of the underlying theory in proposing an informationally efficient covariance estimator. This estimator is then compared to conventional covariance estimates in an empirical application using data from seven asset classes typically considered by institutional investors. Specifically, we show that covariance estimates are sensitive to both truncated estimates involving shorter series and the effects of smoothing which, if left unattended, will produce significantly biased allocations and higher portfolio risk.
Introduction

When historical returns series vary in length, covariances are typically estimated using a shorter common subset of returns thereby discarding some information contained in the longer series. Problems associated with returns truncation are particularly troublesome for allocations across broad asset classes; there are typically only a small number of classes and because many of these are relatively new (e.g., TIPS) they contribute little information to covariance estimates. Covariance precision will necessarily suffer for more severely truncated returns and the information loss from truncation will generally produce inefficient and in some cases biased covariance estimates (Stambaugh, 1997). Perhaps more important are the obvious adverse implications for plan-wide risk management—at a minimum, it is likely that exposures will be miscalculated as will the investor’s overall exposure to risk. We present Monte Carlo evidence below supporting this assertion.

It is also well-known that the reported returns to some asset classes (e.g., Real Estate, Private Equity) are smoothed estimates of the underlying true returns. Smoothing will cause these returns to have artificially lower volatilities and covariations with the remaining asset classes which, if uncorrected, will bias allocations toward the smoothed asset classes. Smoothing is a data problem whose origins lie in the way the returns are computed and reported, e.g., as moving averages of previously observed prices (real estate appraisals) or as a timing issue in which returns are reported at irregular intervals (private equity valuations). In any case, smoothing alters the time series relationships which when combined with truncation may produce seriously misleading covariance estimates and exacerbate exposure to unwanted risk.

In this paper, we draw primarily from Stambaugh (1997) and Fisher and Geltner (2000) to resolve problems associated with returns truncation and smoothing using return streams from
seven asset classes commonly analyzed by institutional investors. We also present results from a Monte Carlo experiment that generates mean-variance optimal portfolios for both cases, i.e., when returns are smoothed and truncated against mean returns and covariances drawn from identical series with the effects of smoothing and truncation removed. The following section develops the covariance model as a response to the above-mentioned considerations. A discussion of the data follows in the third section. In section four, we present our general findings and results from the Monte Carlo experiment. Some concluding remarks follow in the last section.

**Covariance Estimation**

The covariance estimator for returns series of differing length was first introduced by Stambaugh (1997) and the methodology was extended in Pastor and Stambaugh (2002). We summarize and discuss Stambaugh’s derivations in Appendix I at the end of this paper and the reader is referred there for details. The intuition, however, is based on an application of the multivariate normal distribution for which the conditional moments of the distribution of returns to shorter history assets are dependent on moments for the longer lived assets.

Consider, for example, a bivariate case consisting of two assets, J and K, but with J having a longer history. The truncated maximum likelihood estimator (MLE) utilizes the assets’ separate histories to estimate unconditional means and variances but uses the history truncated at K to estimate the covariance. As such, asset K’s moment estimates are not only inefficient as is the covariance estimator but there is no guarantee that the covariance matrix will be positive definite. Stambaugh shows that these estimates can be improved by appealing to the properties of the bivariate normal, i.e., the conditional distribution of asset K (conditional on information contained in asset J returns) has mean and variance that are linear functions of the information
contained in the longer return history. If the return histories are independent, there is no informational gain generated by utilizing the MLE for the conditional distribution of returns. In this case, the conditional distribution yields the truncated estimator.

Although the conditional distribution provides exact solutions for estimators of first and second moments, these estimators implicitly assume that the linear relationships amongst asset returns are constant over time. For example, the conditional mean for asset K in the example above is a linear combination of its unconditional mean and the product of its beta with asset J and asset J’s mean return in the period preceding asset K’s return history—see equation (6)\(^1\). Thus, the unconditional mean is augmented with the information contained in the longer series return history not observed for asset K with the magnitude of this extra information determined by its time-invariant beta. Time invariance holds for conditional second moments as well—see equation (7). In effect, MLE averages the impact of structural changes in the linear relationships among returns series.

There are really two issues here. One is the implied time-invariance which, if overly restrictive, suggests some degree of estimation error which contributes to poor out-of-sample performance. Sample moment estimators, in general, often produce extreme portfolio positions that are inconsistent with “equilibrium” market capitalization weights [Black and Litterman (1992)]. The other issue relates to sample size itself; for very short histories (like TIPS), conditional moment estimates fall victim to a degrees of freedom problem in the unconditional distribution that could translate into less precise estimates of the conditional moments. In these cases, the conditional moments may not be representative of the characteristics of the shorter series. This might be especially relevant in the event that shocks are peculiar to a single series, say, or institutional changes alter the structural relationships among series. Contagion, for

\(^1\) All equation references refer to Appendix I.
example, diffuses across markets and depending on the rate of diffusion will alter the structural relationships between various series but its impact will continue to be averaged with older data. The point is that perceived gains in efficiency depend on being able to extract stable and meaningful linear relationships between series.²

Estimation risk, which complicates this process, arises when sample estimates of parameters of return distributions are implicitly assumed to be the true parameters. Consequent portfolios may be, quite plausibly, inadmissible once estimation risk is explicitly incorporated into the analysis (Klein and Bawa, 1976). More recently, Jorion (1986) and Frost and Savarino (1986) introduce Bayesian estimates of multivariate unconditional returns based on informative priors. Stambaugh shows that the Bayesian predictions (with a diffuse prior), relative to MLE, do not alter estimates of mean returns but scale up MLE covariance estimates due to estimation error. Because the difference between the MLE and Bayesian covariance estimates is shown to be small for portfolios consisting of relatively few assets, we report and discuss MLE only.

In a multivariate world, asset K’s moments are functions of its betas with all other assets of equal or longer returns duration and the conditional maximum likelihood estimators, though still assumed to be time-invariant, utilize information contained in all the longer return histories. The exact multivariate estimation procedure is described below in the results section. We do not address data problems pertaining to missing observations or gaps in returns series but note that these issues are adequately addressed using data-augmentation methods such as the EM algorithm (an iterative MLE method that effectively treats missing data as parameters to be estimated) and Gibbs sampling (to bootstrap the Bayesian pdf).

² We note that time varying estimation schemes are available using Bayesian Dynamic Linear methods in which both means and covariances are recursively updated as new information on returns becomes available, e.g., Kling and Novomestky (1999).
Returns smoothing only complicates attempts to resolve the truncation problem. Both are information problems; truncation throws information away while smoothing filters it. In general, affected series must be unsmoothed beforehand. Working (1960) first commented on the impact that aggregation has on smoothing noting that a random walk, when averaged, induces serial correlation but with an upper bound (0.25). One would expect returns averaging to induce some degree of serial correlation in an otherwise efficient market. Nevertheless, observed levels of correlation, especially for Real Estate and Private Equity returns, appear too high to be explained by simple aggregation. The effects of smoothing are especially well documented in the real estate literature (Geltner (1991, 1993a, 1993b), Quan and Quigley (1991), and Ross and Zisler (1991)) but correlation may also be the consequence of non-synchronous trading (see Campbell, Lo, and MacKinlay (1997) and the references therein), or non-periodic marking-to-market. The smoothing of Real Estate returns is tied largely to the appraisal process in which estimated property values are linear combinations of past subjective appraisals. Similarly for Private Equity, returns are based on subjective valuations of non-publicly traded firms. In both cases, market returns are not observed, i.e., valuations are not tied to a unique market determined price for a single publicly traded security. In Appendix I, we summarize and discuss a method, proposed by Fisher and Geltner (2000), to unsmooth real estate returns in which observed returns are assumed to be an infinite order moving average of past market returns (based on property appraisals). If the moving average process is stationary and invertible, then the unsmoothed returns series can be recovered from the observed lagged, but smoothed, series of returns (see Appendix I for details). We use this method to unsmooth both Real Estate and Private Equity returns in our seven-asset study. The unsmoothing parameters \( a \) for Real Estate and Private
Equity in equation (10) are estimated to be 0.728 (t-stat = 10.5) and 0.436 (t-stat = 4.39) respectively.

Other approaches to unsmooth return series can be found in Shilling (1993) and Wang (2001). These are multivariate approaches; Shilling infers the degree to which the variance in observed returns has been smoothed by exploiting the properties of a biased OLS estimator\(^3\) and its consistent instrumental variables counterpart. Wang, on the other hand, exploits certain cointegrating relationships to get at the degree to which returns variability has been smoothed. We have chosen to apply the Fisher-Geltner algorithm that is developed in Appendix I because of its intuitive appeal and ease of application.

**Data**

The basic building blocks for estimating the covariance matrix are return series for each of the underlying assets. In practice, analysts have several important decisions to make regarding the selection of these basic building blocks. Notably, they will want to choose historical data that is representative of the asset classes that they would consider including in their portfolios. For illustrative purposes, we have chosen common benchmarks for seven asset classes that are frequently considered by institutional investors. The qualitative results of this analysis were robust to the choice of alternative benchmarks for each of these asset classes, the inclusion of additional asset classes, using mixed frequency data, and return series with longer histories. Table 1 lists the return series that were chosen for each asset class and a risk-free rate, series inception date, and number of quarterly observations. Because the Real Estate and Private Equity return series are only provided quarterly, we geometrically linked the higher frequency returns to generate quarterly returns. All analysis in this paper was performed using quarterly

\(^3\) True returns are not directly observed thus creating an errors-in-variables problem which biases OLS estimates.
excess returns, which were calculated by subtracting the risk-free rate from the aforementioned returns. Likewise, all results are presented in quarterly excess return format.

While all seven return series end in December 2004, their inception dates range from February 1997 for TIPS to December 1975 for Fixed Income. The differing inception dates normally force the truncation of longer time series in order to estimate covariances. A set of estimated correlations\(^4\), standard deviations, and mean excess returns for the seven asset classes are presented in Table 2. These are the unconditional covariances based upon the truncated series. We will refer to these as the set of naïve estimates. Again, individual variances are estimated from each series’ history while covariances are truncated to each common historical pairing.

In the Monte Carlo study reported below, we take the investment period to begin in the first quarter 2005 and assume that all managers are mean-variance optimizers who estimate mean returns and covariances using information through 2004, solve the minimum variance portfolio, and hold that portfolio thereafter. We are especially interested in portfolio composition (i.e., possible extreme positions), expected returns, and risk.

**Results**

The naïve covariance estimates, which are equivalent to unconditional MLE for individual returns series but use the truncated series to estimate covariances are presented in Table 2. The set of corrected MLE returns, presented in Table 3, are the product of a two step procedure\(^5\). We first removed the effects of smoothing from Private Equity and Real Estate returns by applying

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\(^4\) Correlations are normalized covariances by dividing the latter through by the product of the two asset classes’ returns standard deviations. This means that variances (covariances between assets J and K where J = K) normalize to unity (diagonal elements in Table I) while off-diagonal (J ≠ K) covariances normalize to correlations ρ, such that \(-1 ≤ ρ ≤ 1\).
the Fisher-Geltner transformation outlined in Appendix I. The covariance matrix was then estimated with a recursive application of Stambaugh’s methodology in the following way:

Return series were organized in descending order of length. The longest—Fixed Income which dates to 1975—was used to estimate its own mean and variance. The next longest series, Real Estate (unsmoothed), was then regressed against Fixed Income and the results were then used to update the conditional mean for Real Estate and its covariance with Fixed Income as given by equations (6)-(7). The returns to Fixed Income and Real Estate then constitute a matrix of “instruments” on which successively shorter returns series are regressed recursively with covariances being revised according to equations (6)-(7). At the end of each recursion, the matrix of instruments is expanded once again. The new matrix of instruments is always truncated to a length equal to the series forming the next longest series, the regression is estimated, and revisions made. That last regression involved TIPS, with a starting date of February 1997, and the matrix of instruments included the remaining six (unsmoothed) return series truncated to this date.

A comparison of Tables 2 and 3 indicate several sign reversals, most notably TIPS (the shortest series), and a scattered few for the remaining series. The results in Table 3 further suggest that TIPS is no longer so strongly and negatively correlated with U.S. Equity, which highlights the informational differential between truncated and longer series. For example, variances and covariances based on short histories of truncated relationships will be overestimated if truncated series correspond to high volatility states. If short and long series are positively (negatively) correlated then the conditional covariance MLE for the short series will be adjusted downward (upward) as indicated by equation (7). Its conditional mean, given by equation (6), will also be adjusted but the direction is uncertain; it will be adjusted downward.

\footnote{All code was programmed in Matlab and is available upon request.}
when the truncated mean on the longer series exceeds its long-run average. This is an especially powerful result that shows how vulnerable covariance estimates are to shorter return series effects in the presence of truncation.

As discussed above, returns smoothing decreases volatility. The standard deviations increase significantly from Table 2 to Table 3 for Private Equity (5.87% to 9.13%) and Real Estate (1.66% to 4.36%). The naive return volatility for TIPS, based on the truncated sample, is likewise revised upward from 2.27% to 3.38%. Other things constant, the naïve portfolio would have over-allocated to these three asset classes and the magnitude of this misallocation rises with volatility underestimation.

Figure 1 maps the efficient frontiers implied by each covariance matrix, using the naive historical mean excess return for each asset class. These indicate quite clearly that the naïve covariance matrix underestimates risk and overestimates returns relative to the more efficient revised covariance matrix, a result consistent with Stambaugh’s (1997) findings. The difference in the locations of the two frontiers is largely due to the underestimated standard deviations of the smoothed return series.

**A Monte Carlo Experiment.** This section reports the results from a Monte Carlo study designed to answer the following two questions: is capital misallocation and portfolio performance significantly affected in the presence of substandard covariance estimates? Though we find that misallocation is indeed significant, and that portfolio performance suffers as a result, the magnitude and cost of misallocation will, in general, depend on the severity of truncation and smoothing and the number of affected assets. We present a simple case with two of seven assets having truncated returns and a third with smoothed returns.
The optimal portfolio corresponds to the vector of asset weights that solves the mean-variance optimization problem using as arguments a vector of mean returns and a covariance matrix. For the following experiment, we use as arguments the corrected parameter estimates given in Table 3 which consist of the unsmoothed Stambaugh-corrected covariance matrix and corresponding corrected mean returns. From these, we generate repeated samples of returns of length equal to 100 (twenty-five years of quarterly returns) for each of the seven asset classes.6

For each generated sample, we estimate the full-information covariance matrix and returns – these estimates are used to solve full-information minimum variance portfolios which are used for baseline comparisons provided in Table 4. We then truncate the simulated series for TIPS and Non-US Equity to 25 and 50 periods respectively7, smooth Real Estate8, and estimate covariances and returns for (1) the truncated and smoothed returns—our naïve model, and (2) then correct the set of naive means and covariances using Stambaugh and the Fisher-Geltner methods outlined above. Separately, these sample mean-variance estimates were then used to solve the minimum variance, long-only, fully-invested portfolio,

$$\min \left( \frac{1}{2} w'Vw \right) \quad \text{s.t.} \sum_{i=1}^{7} w_i = 1, \quad w_i \geq 0$$

where $V$ is case specific, i.e., it corresponds to either the full-information, naïve, or corrected covariance matrix.

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6 Specifically, if the estimated covariance matrix $V$ is positive definite, then it will have a Cholesky decomposition—a lower triangular square matrix $A$ such that $V = AA'$. Then, for any vector of mean returns $\mu$, a single realization of a 7 x 1 vector of simulated returns, with covariance structure given by $V$, can be generated from $\mu + Ae$ where $e$ is a 7 x 1 vector of standard normal random variables. Repeating this process over $t = 1,...,100$ generates a 7 x 100 sample of returns with mean $E(\mu + Ae) = \mu$ and covariance $E(Aee' A') = V$ since $E(ee')$ is the identity matrix.

7 These match the historical series lengths for TIPS and Non-US equity relative to Fixed Income.

8 The choice of smoothing parameter, $\alpha$, is 0.5 and was selected to generate a smoothed series that understates volatility by about one-half (empirically, we estimate this parameter in the range $0.436 - 0.728$ for Private Equity and Real Estate). Returns are smoothed according to $r_{s,t} = \alpha r_{s,t-1} + \left( 1 - \alpha \right) r_t$ where $r_{s,t}$ is the smoothed return and $r_t$ the actual return. The simulation program is available upon request.
The misallocation question was approached by measuring the sum of the *absolute* differences between the full-information weight vector in case (1) and the weight vectors in cases (2) and (3), i.e., we are interested in the distributions of the statistic:

\[
z_{n,j} = \sum_{i=1}^{7} |w^* - w_{i,j}| \quad \text{for } n = 1, \ldots, N, \quad j = 1, 2
\]

where \(w^*\) denotes the full-information optimal portfolio, \(w_j\) the optimal portfolio associated with either the naïve or corrected moments estimates, and \(N\) the number of replications in the experiment. We generate \(N\) matrices of returns each of size 100 x 7. Each matrix is then used to estimate expected returns and covariances for each of the three cases defined above and these moment estimates solve three mean-variance efficient portfolios. Therefore, \(z_n\) measures the absolute amount of misweighting between the full-information portfolio and each of the portfolios based on the naïve and corrected covariance estimates respectively.

The weights in the summation above are matched pairs, by asset class, and their sum measures the aggregate deviation from the optimal portfolio weighting. The long-only constraint and absolute differences place an upper bound on the maximum misallocation equal to 2\(^9\). Thus, the statistic \(z_n\), when divided by 2, measures the amount of misallocation, in percent, relative to the maximum allocation loss. Of primary interest is testing for relative misallocation, i.e., whether \(z_n\) is higher for the naïve model relative to the corrected covariance model. To this end, we drew \(N = 10,000\) samples of returns using the Monte Carlo method just described. A simple test of the null hypothesis that this statistic has zero mean is a parametric t-test on the coefficient in a regression of the difference in the paired returns on a constant. These t-statistics are

\[^9\] Consider, for example, a two asset problem. Maximum misallocation occurs when the two weight vectors are orthogonal, i.e., \(w_1 = (0,1)\) and \(w_2 = (1,0)\) whose sum of absolute differences is equal to 2.
supplemented with the nonparametric Wilcoxon test for matched pairs\textsuperscript{10}. Relevant statistics and test results are presented in Table 4. The Wilcoxon test is applied to the same three sets of matched pairs—the differences between the full-information portfolio and each suboptimal portfolio (i.e., the naïve and corrected portfolios) and a third measuring the difference between the naïve and corrected portfolios. Wilcoxon is a test of medians; specifically, that the median of the differenced distribution (the paired differences) is zero, i.e., the two returns distributions share a common central tendency.

Panel A of Table 4 shows that the average misallocation for the naïve portfolio is 28.66% and the test of Wilcoxon easily rejects the null that they share a common median. Utilizing the Stambaugh and Fisher-Geltner corrections reduces misallocation to 20.52% which is about 8.14% less but interestingly, the test of Wilcoxon cannot reject the null that the corrected and full information weight distributions share the same median. Clearly, though, the mean misallocation difference between the naïve and corrected covariance portfolios is significant and, in this case, about 8.14% on average in each quarter.

Portfolio performance measures are straightforward computations using expected returns, which are the product of the mean-variance optimal weights and mean returns specific to each case \((w'r)\), and portfolio risk \((w'Vw)^{1/2}\), where again, \(w\) is the case specific mean-variance optimal weight vector and \(V\) is the case specific estimated covariance matrix. Results, which are presented in Panel B of Table 4 show that misallocation results in more extreme positions as evidenced by lower portfolio returns and higher portfolio risk. Indeed, the median quarterly return under the naïve approach (0.787%) underperforms both the optimal return (0.873%) by 0.086% per quarter and the corrected covariance case (0.868%) by a little over 0.081% per quarter.

\textsuperscript{10} Since \(z_n\) is a sum of absolute differences, its distribution is not necessarily normal; indeed, it is positive and skewed right. The nonparametric tests are distribution free and are reported as a check on the robustness of our results.
quarter. The naïve portfolio’s tendency to take extreme positions can be seen in the volatility of its returns; 2.51% per quarter relative to 2.16% and 2.22% for the full-information and corrected cases respectively as well as the lowest Sharpe ratio (measured over the 10,000 replications).

The long-only constraint places bounds on extreme positions. Nevertheless, the naïve portfolios contain approximately 13% more extreme positions (zero positions on an asset class) while portfolios generated from the corrected moment estimates produce roughly 5% more zero positions relative to the optimal portfolio. When the long-only constraint is removed, extreme positions are accentuated but only for the naïve case.

The naïve model’s rather significant tendency to bias portfolio weightings generates increased risk primarily because it misallocates capital to the wrong asset classes. Bias, itself, tends to favor the smoothed series and, to the extent that a truncated series is currently experiencing a period of low volatility, bias over-weights those assets as well. As a result, the naïve model will produce under-stated value-at-risk estimates which may lead managers to take on bets that are not otherwise supported by the data. The magnitude of the weighting bias is a function of both the proportion of assets with limited histories and their relative series lengths—shorter histories will exacerbate the bias as will the magnitude of smoothing.

Conclusions

The fundamental premise of modern portfolio theory is that risk can be measured, targeted, minimized, and otherwise managed. Most challenging in this premise is measurement. The covariance matrix presented in this paper is the outcome of considerable empirical effort. We have adjusted what would otherwise be considered standard methods for measuring covariances for the smoothing of returns (Real Estate and Private Equity), and short return histories. These adjustments remove potential biases and improve efficiency by maximizing the information
content of our estimates. Finally, we explicitly address the shortcomings associated with
covariance estimation using smoothed and truncated series. Specifically, we examine the risk
and allocative consequences from a Monte Carlo experiment and conclude that the naïve
covariance estimates generate significant weight bias, undesirable allocative tilts, and higher
portfolio risk.

Further research might be directed at improved unsmoothing algorithms, especially for
non-real estate classes such as private equity and hedge funds (hedge funds are not examined in
this paper). And that research should examine more closely competing theories of smoothing.
Attention, too, to the time-invariance property may also be productive, e.g., covariance
relationships among short and long series may be more efficiently estimated using recursive
methods that update each time period (e.g., Kalman filtering). Nevertheless, the applications
presented in this paper clearly improve allocative efficiency while illuminating some of the
deficiencies associated with conventional covariance estimates.
Appendix I – Derivation of Stambaugh & Fisher-Geltner Methods

Stambaugh’s Covariance Matrix. Consider the simplest case with two return series \( r_{1,t} \) for \( t = 1,\ldots,T \) and \( r_{2,t} \) for \( t = 1,\ldots,S \) where \( S < T \). Truncation to \( S \) would imply that means be equal to their maximum likelihood estimates:

\[
E \begin{bmatrix} r_{1,t} \\ r_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}.
\]

(1)

Here, \( \mu_1 \) and \( \mu_2 \) are the sample MLE for the truncated sample of size \( S \). Similarly, the MLE covariance matrix based upon \( S \) is equal to:

\[
V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}
\]

(2)

where \( V_{ij} \) are scalars. The parameters \( \mu_1 \) and \( V_{11} \) are typically replaced with their MLE counterparts based on the full sample of size \( T \).

Consider now the likelihood function for these series which can be written as a combination of the information unique to \( r_{1,t} \) (i.e., the observations in \( r_1 \) not common to \( r_2 \)) and the information common to both series, e.g. the joint likelihood given by:

\[
P(r_{1,t}, r_{2,s} | E, V) = \prod_{t=s+1}^{T} \left[ (2\pi)^{-1/2} |V_{11}|^{-1/2} \exp\left\{-\frac{1}{2} (r_{1,t} - \mu_1) V_{11}^{-1} (r_{1,t} - \mu_1) \right\} \right] \\
\times \prod_{t=1}^{s} \left[ (2\pi)^{-1} |V|^{-1/2} \exp\left\{-\frac{1}{2} (r - E) V^{-1} (r - E) \right\} \right].
\]

(3)

The second part of the likelihood is the distribution of joint information while the first part describes the contribution to the likelihood function from the information unique to \( r_{1,t} \). The truncated estimator ignores the second half of the likelihood function. Ordinarily, one would maximize this likelihood with respect to \( E \) and \( V \) but unfortunately these two sets of parameters
do not appear in both halves of the likelihood. Stambaugh uses a result from Anderson (1957) that rewrites the joint likelihood as the product of a marginal $P(r_{1,t})$ and a conditional density $P(r_{2,t}|r_{1,t})$. Assuming returns to be multivariate normal, we get the familiar result that the conditional mean of $r_{2,s}$ (conditional on $r_{1,s}$) is equal to:

$$\mu_2 = V_{22}^{-1} V_{21} (r_{1,t} - \mu_1)$$

(4)

where $V_{22}^{-1} V_{21}$ is the regression coefficient of $r_{2,s}$ on $r_{1,s}$. If $r_{2,t}$ and $r_{1,t}$ are positively correlated ($\beta > 0$) and a return $r_{1,t}$ is observed to be above its mean $\mu_1$, then $\mu_2$ is adjusted downward from its unconditional value. (This is what statisticians mean by “regression toward the mean”.) Furthermore, the conditional covariance given by the multivariate normal is:

$$V_{22} - V_{21} V_{11}^{-1} V_{12}.$$  

(5)

In the more general multivariate case, $r_{1,t}$ is a vector of returns on $N_1$ assets all with $T > S$ observations and $r_{2,t}$ is an $N_2$ vector of returns on the shorter time series. In that case, $\mu_1$ and $\mu_2$ are mean return vectors of size $N_1$ and $N_2$ respectively so that $E$ is an $N = N_1 + N_2$ vector, $V_{11}$ is $(N_1 \times N_1)$, $V_{22}$ is $(N_2 \times N_2)$, $V_{12}$ is $(N_1 \times N_2)$ and $V_{21} = V_{12}'$. Furthermore, $V_{21} V_{11}^{-1} = \beta$ is now an $(N_2 \times N_1)$ matrix of regression coefficients.

The objective is to derive estimates for these covariance matrices. To estimate the maximum likelihood estimators for the moments of the conditional density, regress $r_{2,t}$ on $r_{1,t}$ using $S$ observations saving the covariance matrix of the residuals as $\hat{\Sigma}$. Likewise, estimate mean returns for $\mu_1$ and $\mu_2$ using $T$ and $S$ observations respectively. Then, applying the results in (3) and (4), adjust the truncated mean for $r_{2,t}$ given in (1) by conditioning on the information in $r_{1,t}$.
\[ \hat{\mu}_2 = \hat{\mu}_{2,S} + \hat{\beta} (\hat{\mu}_{1,T} - \hat{\mu}_{1,S}) . \quad (6) \]

Therefore, if the two returns series are positively correlated and the mean of the longer series exceeds its truncated mean, the mean for the shorter series is adjusted upward. That is, the truncated mean is most likely biased downward. Likewise, the truncated covariance matrices are adjusted according to (see Stambaugh):

\[ \hat{V}_{22} = \hat{\Sigma} + \hat{\beta} \hat{V}_{11} \hat{\beta}^\prime; \quad \hat{\Sigma} = (\hat{V}_{22,s} - \hat{\beta} \hat{V}_{11,s} \hat{\beta}) , \quad \hat{\beta} = V_{11,s}^{-1} V_{11,r}^{-1} \quad thus \]
\[ \hat{V}_{22} = \hat{V}_{22,s} - \hat{\beta} (\hat{V}_{11,s} - \hat{V}_{11}) \hat{\beta} \]
\[ \hat{V}_{21} = \hat{\beta} \hat{V}_{11} ; \]
\[ \hat{V}_{21} = \hat{V}_{21,s} - \hat{\beta} (\hat{V}_{11,s} - \hat{V}_{11}) \]
\[ (7) \]

It is easy to show that \( \hat{\beta} \hat{V}_{11} \hat{\beta}^\prime \) is identical to \( V_{21} V_{11}^{-1} V_{12} \) in equation (5). In equation (7), it is also true that the covariance between \( r_{1,t} \) and \( r_{2,t} \) is a linear rescaling of the covariation in \( r_{1,t} \) with the magnitude depending on the strength of their covariation. Moreover, revisions to \( \hat{V}_{22} , \hat{V}_{21} \) depend on the how much the covariation in the longer series changes over the time interval \( T-S \).

**Removing the effects of smoothing alá Fisher-Geltner.** Consider, for example, the appraised value \( P \) which is a moving average of current and past comps \( P_{t-i}^* \):

\[ P_t = w_0 P_t^* + w_1 P_{t-1}^* + w_2 P_{t-2}^* + ... \]
\[ \sum_{i=0}^{\infty} w_i = 1 \]
\[ (8) \]

Rewrite the weights to be geometrically declining such that \( w_i = (1-w_0)^i w_0 \) for some scalar value
of $w_0 < 1$. Let $a = 1-w_0$. Note as well that $a = w_1/w_0$. Substituting into (8) yields:

$$P_t = w_0 P^*_t + aw_0 P^*_{t-1} + a^2 w_0 P^*_{t-2} + ...$$

Now, replace $P$ with its natural logarithm, lag one period and subtract the resulting expression from (8) yielding a moving average of returns:

$$r_t = w_0 r^*_t + aw_0 r^*_{t-1} + a^2 w_0 r^*_{t-2} + ...$$

(9)

where $r_t = \ln(P_t)-\ln(P_{t-1})$. Obviously, returns are a weighted average of past market rates of return and this is the source of the smoothing. Divide both sides of (9) by $w_0$. The resulting expression is the solution to a first order difference equation in $r_t/w_0$. Expressing (9) as an AR(1) process for $r_t$ and substituting for $a$ yields:

$$r_t = w_0 r^*_t + ar_{t-1}$$

(10)

We seek the unsmoothed component $r^*_t$ which is:

$$r^*_t = r_t - \left(\frac{w_1}{w_0}\right) r_{t-1}. \quad (11)$$

Thus, we use the observed smoothed returns to recover the “market” return. An estimator for $w_1/w_0$ can be obtained from an OLS regression of observed $r_t$ on its lagged value. This is a special case of an ARMA model for which the moving average component, under certain stationarity conditions, is invertible. An invertible infinite order moving average is equivalent to
a first order autoregressive (AR) model whose parameter is estimated using ordinary least squares. The series can then be unsmoothed and covariances estimated thereafter.
REFERENCES


## Table 1. Return Series Descriptions (All Data Ends Dec-04)

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Inception</th>
<th>Return Series Description</th>
<th>Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIPS</td>
<td>Feb-97</td>
<td>Lehman TIPS All Maturities</td>
<td>31</td>
</tr>
<tr>
<td>Non-US Equity</td>
<td>Jun-89</td>
<td>Citigroup Global BMI ex-U.S.</td>
<td>62</td>
</tr>
<tr>
<td>High Yield</td>
<td>Aug-86</td>
<td>Merrill Lynch High Yield Master II</td>
<td>73</td>
</tr>
<tr>
<td>Private Equity</td>
<td>Dec-83</td>
<td>Venture Economics All Priv. Eq.†</td>
<td>84</td>
</tr>
<tr>
<td>US Equity</td>
<td>Dec-78</td>
<td>Frank Russell 3000</td>
<td>104</td>
</tr>
<tr>
<td>Real Estate</td>
<td>Dec-77</td>
<td>NCREIF NPI</td>
<td>108</td>
</tr>
<tr>
<td>Fixed Income</td>
<td>Dec-75</td>
<td>Lehman Aggregate</td>
<td>116</td>
</tr>
<tr>
<td>Risk-Freee</td>
<td>Dec-25</td>
<td>Ibbotson U.S. 30-day T-Bill</td>
<td>n.a.</td>
</tr>
</tbody>
</table>

† Returns prior to Mar-84 were censored due to lack of representation
### Table 2. Naïve Correlation Matrix, Standard Deviations, and Mean Returns (Quarterly)

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>TIPS</th>
<th>Non-US Equity</th>
<th>High Yield</th>
<th>Private Equity</th>
<th>US Equity</th>
<th>Real Estate</th>
<th>Fixed Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIPS</td>
<td>1</td>
<td>-0.42</td>
<td>-0.15</td>
<td>-0.55</td>
<td>-0.64</td>
<td>-0.13</td>
<td>0.36</td>
</tr>
<tr>
<td>Non-US Equity</td>
<td>-0.42</td>
<td>1</td>
<td>0.46</td>
<td>0.55</td>
<td>0.78</td>
<td>0.13</td>
<td>-0.02</td>
</tr>
<tr>
<td>High Yield</td>
<td>-0.15</td>
<td>0.46</td>
<td>1</td>
<td>0.24</td>
<td>0.56</td>
<td>-0.10</td>
<td>0.13</td>
</tr>
<tr>
<td>Private Equity</td>
<td>-0.55</td>
<td>0.55</td>
<td>0.24</td>
<td>1</td>
<td>0.60</td>
<td>0.18</td>
<td>-0.17</td>
</tr>
<tr>
<td>US Equity</td>
<td>-0.64</td>
<td>0.78</td>
<td>0.56</td>
<td>0.60</td>
<td>1</td>
<td>0.00</td>
<td>0.21</td>
</tr>
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<td>Real Estate</td>
<td>-0.13</td>
<td>0.13</td>
<td>-0.10</td>
<td>0.18</td>
<td>0.00</td>
<td>1</td>
<td>-0.29</td>
</tr>
<tr>
<td>Fixed Income</td>
<td>0.36</td>
<td>-0.02</td>
<td>0.13</td>
<td>-0.17</td>
<td>0.21</td>
<td>-0.29</td>
<td>1</td>
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<tr>
<td>Standard Deviations (%)</td>
<td>2.27</td>
<td>9.10</td>
<td>3.89</td>
<td>5.87</td>
<td>8.33</td>
<td>1.66</td>
<td>3.51</td>
</tr>
<tr>
<td>Mean Returns (%)</td>
<td>1.15</td>
<td>0.85</td>
<td>1.16</td>
<td>1.77</td>
<td>2.06</td>
<td>0.60</td>
<td>0.72</td>
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</table>

### Table 3. Corrected Correlation Matrix, Standard Deviations and Mean Returns (Quarterly)

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>TIPS</th>
<th>Non-US Equity</th>
<th>High Yield</th>
<th>Private Equity</th>
<th>US Equity</th>
<th>Real Estate</th>
<th>Fixed Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIPS</td>
<td>1</td>
<td>0.19</td>
<td>0.44</td>
<td>-0.25</td>
<td>0.06</td>
<td>-0.23</td>
<td>0.89</td>
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<tr>
<td>Non-US Equity</td>
<td>0.19</td>
<td>1</td>
<td>0.48</td>
<td>0.52</td>
<td>0.82</td>
<td>0.09</td>
<td>0.22</td>
</tr>
<tr>
<td>High Yield</td>
<td>0.44</td>
<td>0.48</td>
<td>1</td>
<td>0.21</td>
<td>0.61</td>
<td>0.03</td>
<td>0.51</td>
</tr>
<tr>
<td>Private Equity</td>
<td>-0.25</td>
<td>0.52</td>
<td>0.21</td>
<td>1</td>
<td>0.64</td>
<td>0.13</td>
<td>-0.22</td>
</tr>
<tr>
<td>US Equity</td>
<td>0.06</td>
<td>0.82</td>
<td>0.61</td>
<td>0.64</td>
<td>1</td>
<td>0.03</td>
<td>0.19</td>
</tr>
<tr>
<td>Real Estate</td>
<td>-0.23</td>
<td>0.09</td>
<td>0.03</td>
<td>0.13</td>
<td>0.03</td>
<td>1</td>
<td>-0.25</td>
</tr>
<tr>
<td>Fixed Income</td>
<td>0.89</td>
<td>0.22</td>
<td>0.51</td>
<td>-0.22</td>
<td>0.19</td>
<td>-0.25</td>
<td>1</td>
</tr>
<tr>
<td>Mean Returns (%)</td>
<td>0.92</td>
<td>0.83</td>
<td>1.12</td>
<td>2.07</td>
<td>2.05</td>
<td>0.61</td>
<td>0.72</td>
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</tbody>
</table>
Table 4. Monte Carlo Simulation Results

<table>
<thead>
<tr>
<th>Panel A - Misallocation&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Mean (%)</th>
<th>t-statistic</th>
<th>Wilcoxon&lt;sup&gt;c&lt;/sup&gt;</th>
<th>Wilcoxon p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naïve</td>
<td>28.66</td>
<td>391.87</td>
<td>-31.57</td>
<td>(p = 0.000)</td>
</tr>
<tr>
<td>Corrected</td>
<td>20.52</td>
<td>203.28</td>
<td>-0.41</td>
<td>(p = 0.680)</td>
</tr>
<tr>
<td>Naïve - Corrected&lt;sup&gt;b&lt;/sup&gt;</td>
<td>8.14</td>
<td>64.51</td>
<td>-29.31</td>
<td>(p = 0.000)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B - Performance&lt;sup&gt;d&lt;/sup&gt;</th>
<th>Mean (%)</th>
<th>Median (%)</th>
<th>Risk (%)</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full-information optimal</td>
<td>0.873</td>
<td>0.873</td>
<td>2.169</td>
<td>0.404</td>
</tr>
<tr>
<td>Naïve</td>
<td>0.787</td>
<td>0.759</td>
<td>2.510</td>
<td>0.313</td>
</tr>
<tr>
<td>Corrected</td>
<td>0.868</td>
<td>0.848</td>
<td>2.221</td>
<td>0.391</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Return Differential</th>
<th>Mean (%)</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full-information - Naïve</td>
<td>0.086</td>
<td>44.365</td>
</tr>
<tr>
<td>Full-information - Corrected</td>
<td>0.005</td>
<td>4.052</td>
</tr>
<tr>
<td>Corrected - Naïve</td>
<td>0.081</td>
<td>43.153</td>
</tr>
</tbody>
</table>

<sup>a</sup> Mean absolute weight differential relative to full-information portfolio

<sup>b</sup> Naïve-corrected mean absolute weight differential

<sup>c</sup> Ho: optimal minus stated weight vector has zero median

<sup>d</sup> Quarterly performance in percent
Figure 1. Efficient Frontier

Portfolio Risk (%)

Portfolio Return (%)

- Naïve
- Corrected